IMSc/2001/08/48

Semi-classical States in the Context of Constrained Systems

G. Date *

The Institute of Mathematical Sciences, CIT Campus, Chennai-600 113, INDIA.

Parampreet Singh †

Inter-University Centre for Astronomy and Astrophysics,

Post Bag 4, Ganeshkhind, Pune-411 007, INDIA.

Abstract

Algebraic quantization scheme has been proposed as an extension of the Dirac quantization scheme for constrained systems. Semi-classical states for constrained systems is also an independent and important issue, particularly in the context of quantum geometry. In this work we explore this issue within the framework of algebraic quantization scheme by means of simple explicit examples. We obtain semi-classical states as suitable coherent states a la Perelomov. Remarks on possible generalizations are also included.

PACS numbers: 03.65.Ca, 03.65.Fd

Typeset using REVTEX

*e-mail: shyam@imsc.ernet.in

†e-mail: param@iucaa.ernet.in

I. INTRODUCTION

Dirac's procedure of quantizing a classical theory with first class constraints consists of several steps. Firstly one quantizes the system ignoring the constraints to get a *kinematical* Hilbert space, $\mathcal{H}_{kin}$. The constraints, represented as self-adjoint operators on $\mathcal{H}_{kin}$ are then imposed as operator equations and physical states are defined to be the kernel of the constraint operators. The implicit assumption that physical states belong to the $\mathcal{H}_{kin}$ turns out to be wrong in many cases of interests and hence a refinement is proposed via the so called (refined) Algebraic Quantization Scheme [1–6]. Essentially this includes a 'rigging' of $\mathcal{H}_{kin}$, $\Omega \subset \mathcal{H}_{kin} \subset \Omega^*$ and physical states are sought in Ω^*. This allows physical states to be 'distributional' and also allows new physical inner product to be chosen to define $\mathcal{H}_{phy}$ and physical observables. A map $\eta \ : \ \Omega \ \longrightarrow \ \Omega^*$ plays a central role. An example of such a map is provided by the so called 'group averaging procedure'.

There is an independent issue of semi-classical states for a quantum system. The canonical example of 'harmonic oscillator coherent states' (standard coherent states), eigenstates of the annihilation operators, embodies the idea of semi-classical states. These states are labeled by points in the classical phase space Γ. Furthermore there are observables (positions and momenta) with respect to which these states are 'peaked' at points in the phase space. This is particularly easy when the phase space is R^{2N}, since the generalized eigenvalues of the positions and momenta operators provide global coordinates for the phase space. This is clearly not possible when the phase space is topologically non-trivial. Such a phase space can typically be obtained as reduced phase spaces, $\hat{\Gamma}$ - Constrained surface modulo orbits of the constraints, and one needs a suitable generalization of the notion of semi-classical states.

Clearly, the first property one needs is that the semi-classical states (in the quantum

Hilbert space) be labeled by points of a classical phase space, i.e. $|\omega\rangle, \omega \in \Gamma$. Second property needed is that of 'peaking'. Given any quantum observable $\hat{F}$, one can immediately get a function on the classical phase space, $f(\omega) \equiv \langle\omega|\hat{F}|\omega\rangle/\langle\omega|\omega\rangle$. The idea of peaking is that we can find enough observables $\hat{F}_i$ such that specifying $f_i(\omega) = C_i$ will enable one to obtain a unique point, $\omega(C_i)$, in the phase space. Of course there will be fluctuations: $\Delta f^2(\omega) = [f^2](\omega) - ([f][(\omega))^2$. These are to be 'small' in a suitable sense eg. 'minimum' or within specified windows $\pm|\delta C_i|$. Clearly one must have at least 2N such observables. If we can find such $|\omega\rangle$ and $\hat{F}_i$, then we say that $|\omega\rangle$ are candidate semi-classical states.

Notice that the notion of semi-classical states itself does *not* require any approximation or limiting procedure ($\hbar \to 0$, large quantum numbers etc). These are just states corresponding to classical states, thus incorporating correspondence principle. In principle there could be two or more distinct sets of semi-classical states. These could be labeled by same phase space or different phase spaces. The latter case may be construed as an example of potentially equivalent quantum theories corresponding to two different classical systems. A requirement that a quantum theory admits such semi-classical states is a non-trivial requirement as an arbitrarily constructed Hilbert space may or may not admit $|\omega\rangle, \hat{F}_i$ for any choice of a classical phase space. Whether such a notion of semi-classical states is too permissive or too restrictive is not clear at present.

For constrained systems, in practice, it is often convenient to follow the Dirac quantization procedure (as opposed to the reduced phase space quantization). The notion of semi-classical states should now be properly defined in the $\mathcal{H}_{phy}$ and with respect to *physical* observables. One may not have as much much control over $\mathcal{H}_{phy}$ as over $\mathcal{H}_{kin}$ as is the case at present with quantum geometry. One could try to define semi-classical states in $\mathcal{H}_{phy}$ by first defining them in $\mathcal{H}_{kin}$ and performing group averaging on them. The peaking property however still needs to be specified in $\mathcal{H}_{phy}$ using physical observables. Alternatively one should obtain a relation between peaking defined relative to physical quantities and relative to kinematical quantities.

We explore such a strategy in the context of simple toy models with a single constraint. For the class of models for which physical observables contain a Lie algebra, one can use corresponding generalized coherent states a la Perelomov [7] as candidate semi-classical states. Furthermore expectation values of physical observables in $\mathcal{H}_{phy}$ can be computed in $\mathcal{H}_{kin}$.

The paper is organized as follows:

Section II gives a schematic (formal) derivation of the main result.

Section III discusses explicit examples implementing the schematic derivation. The examples are with $\Gamma = R^4$ and a single quadratic constraint. This has three cases involving compact and non-compact semi-simple groups.

Section IV contains remarks on further examples and generalization. A discussion of results, possible extensions is also included.

II. GENERAL SCHEME

Let ϕ denote a single constraint (a self adjoint operator on $\mathcal{H}_{kin}$) and let G be a group commuting with the constraint. Let $|\xi, k\rangle$ denote group coherent states labeled by ξ and constructed from an irreducible representation of G labeled by k. ξ typically denotes points in a coset space while k can be a multi-index in general. Clearly $\mathcal{H}_{kin}$ carries a representation, in general reducible, of the group and ϕ is a multiple of identity on each of the irreducible blocks. Clearly, the constraint will have a well defined value on every irreducible block. Specific value of the constraint will thus select particular irreducible representation (and possibly copies thereof) labeled by , say, $\widetilde{k}$.

In $\mathcal{H}_{kin}$ we have a resolution of identity in the form,

$$\int dk \int d\mu_\xi \, |\xi, k\rangle\langle\xi, k| = 1. \tag{1}$$

The integration over k (which can be a sum if k takes discrete values) is over those values which occur in representations of G in $\mathcal{H}_{kin}$ and $d\mu_\xi$ is a group invariant measure on a coset space.

Following the algebraic quantization scheme, let Ω be a suitable dense subspace of $\mathcal{H}_{kin}$ so that we obtain a rigging: $\Omega \subset \mathcal{H}_{kin} \subset \Omega^*$. For every $|\psi\rangle \in \Omega$ we have,

$$|\psi\rangle = \int dk \int d\mu_\xi \, \langle\xi, k|\psi\rangle \, |\xi, k\rangle. \tag{2}$$

A map $\eta : \Omega \to \Omega^*$ is proposed to be provided by group averaging so that

$$(\psi| = \frac{1}{V} \int d\lambda \int dk \int d\mu_\xi \, \langle\psi|\xi, k\rangle\langle\xi, k| \, e^{-i\lambda\hat{\phi}} \tag{3}$$

where V is the group volume, suitably regulated if necessary. We denote elements of Ω^* generically by $(\cdot|$ (round bra instead of angular bra). Now,

$$\frac{1}{V} \int d\lambda \, \langle\xi, k| \, e^{-i\lambda\hat{\phi}} = \delta(k - \widetilde{k})(\xi, \widetilde{k}|. \tag{4}$$

Hence we get,

$$(\psi| = \int d\mu_\xi \, \langle\psi|\xi, \widetilde{k}\rangle \, (\xi, \widetilde{k}|. \tag{5}$$

The physical inner product, denoted as $\langle , \rangle$, is defined as [2–6]

$$\langle\eta\,\psi', \eta\,\psi\rangle_{phy} = (\psi|\psi'\rangle. \tag{6}$$

The inner product evaluates to

$$
\begin{aligned}
(\psi|\psi') &= \int dk' \int d\mu_{\xi'} \int d\mu_\xi \, \langle\psi|\xi,\widetilde{k}\rangle \, (\xi,\widetilde{k}|\xi',k') \, \langle\xi',k'|\psi'\rangle \\
&= \int d\mu_{\xi'} \int d\mu_\xi \, \langle\psi|\xi,\widetilde{k}\rangle \, \langle\xi,\widetilde{k}|\xi',\widetilde{k}\rangle \, \langle\xi',\widetilde{k}|\psi'\rangle.
\end{aligned} \tag{7}
$$

In the first line, we have used resolution of identity on $|\psi'\rangle$. Then we use equation (4) and the fact that the constraint operator is 'block diagonal' with respect to the resolution of identity, to get to the next line involving only the inner product in $\mathcal{H}_{kin}$.

Similarly, the expectation value of a physical observable $\widehat{A}$ is defined as

$$
\langle\eta\,\psi',\widehat{A}\,\eta\,\psi\rangle = \langle\widehat{A}\,\eta\,\psi',\eta\,\psi\rangle = \langle\eta\,\widehat{A}\,\psi',\eta\,\psi\rangle = (\psi|\widehat{A}\,\psi') \tag{8}
$$

which evaluates to

$$
\begin{aligned}
(\psi|\widehat{A}\psi') &= \int d\mu_\xi \int d\mu_{\xi'} \langle\psi|\xi,\widetilde{k}\rangle \, \langle\xi,\widetilde{k}|\xi',\widetilde{k}\rangle \, \langle\xi',\widetilde{k}|\widehat{A}\,\psi'\rangle \\
&= \int dk \int d\mu_\xi \int d\mu_{\xi'} \int d\mu_{\xi''} \langle\psi|\xi,\widetilde{k}\rangle \, \langle\xi,\widetilde{k}|\xi',\widetilde{k}\rangle \, \langle\xi',\widetilde{k}|\widehat{A}|\,\xi'',k\rangle \, \langle\xi'',k|\psi'\rangle.
\end{aligned}
$$

Note that for $|\psi\rangle \in \Omega$, the resolution of identity involves various representations and thus some of the integrals over the coherent states labels survive. If however, the kinematical states are chosen as $|\psi\rangle = |\xi_o,\widetilde{k}\rangle$, $|\psi'\rangle = |\xi_o^{'},\widetilde{k}\rangle$, then these integrals can be done. For these choices, the inner product becomes

$$
\langle\eta\,\psi'\,,\eta\,\psi\rangle = \int d\mu_{\xi'} \int d\mu_\xi \, \langle\xi_o,\widetilde{k}|\xi,\widetilde{k}\rangle \, \langle\xi,\widetilde{k}|\xi',\widetilde{k}\rangle \, \langle\xi',\widetilde{k}|\xi_o^{'},\widetilde{k}\rangle, \tag{9}
$$

which on using the resolution of identity within an irreducible representation becomes

$$
\langle\eta\,\psi'\,,\eta\,\psi\rangle = \langle\xi_o,\widetilde{k}|\xi_o^{'},\widetilde{k}\rangle = \langle\psi|\psi'\rangle\,. \tag{10}
$$

This is the statement that if kinematical states are chosen as the coherent states of the Lie group generated by a subset of physical observables with the representation index selected by the constraint, then the physical inner product for the corresponding states is same as the kinematical inner product.

The matrix elements of physical observables for the same choice of states also simplifies in a similar manner and becomes,

$$
\begin{aligned}
(\psi|\widehat{A}\psi'\rangle &= \int dk \int d\mu_{\xi} \int d\mu_{\xi'} \int d\mu_{\xi''} \langle\psi|\xi,\widetilde{k}\rangle\, \langle\xi,\widetilde{k}|\xi',\widetilde{k}\rangle\, \langle\xi',\widetilde{k}|\widehat{A}|\,\xi'',k\rangle\, \langle\xi'',k|\psi'\rangle \\
&= \int d\mu_{\xi} \int d\mu_{\xi'} \langle\xi_0,\widetilde{k}|\xi,\widetilde{k}\rangle\, \langle\xi,\widetilde{k}|\widehat{A}|\xi',\widetilde{k}\rangle\, \langle\xi',\widetilde{k}|\xi_0',\widetilde{k}\rangle \\
&= \langle\xi_o,\widetilde{k}|\widehat{A}|\xi_o',\widetilde{k}\rangle \;=\; \langle\psi|\widehat{A}|\psi'\rangle\,. \qquad (11)
\end{aligned}
$$

Evidently, the fluctuations in physical observables in $\mathcal{H}_{kin}$ and $\mathcal{H}_{phy}$ are related as

$$
(\Delta\widehat{A}^2)_{phy} = (\Delta\widehat{A}^2)_{kin}. \qquad (12)
$$

Eqs. (10-12) are the key results. The observation is that *if one uses the coherent states of a selected representation of the Lie group generated by a subset of the physical observables, then the inner product, expectation values of and the quantum fluctuations in physical observables, computed with reference to these are identical whether computed in the $\mathcal{H}_{phy}$ or $\mathcal{H}_{kin}$.* Several remarks are in order.

Remarks:

(1) There is no mention of semi-classical states in the above. The result is strictly a property of coherent states. Even here properties really used are the resolution of identity, labeling of coherent states by some coset space, coherent states being constructed per irreducible representation. In particular, peaking property is not referred to. Group averaging is also used only to the extent that it selects a particular irreducible representation. Although group averaging behaves as though a projection operator, it does *not* give a state in $\mathcal{H}_{kin}$ in general. In particular we do *not* assume that group average of a coherent state belongs to the $\mathcal{H}_{kin}$. In the final expressions we did assume that a selected representation is contained in Ω. Constraint is of course used to admit a group G whose coherent states have been used.

Semi-classical states can now be introduced with the further assumption namely *the reduced phase space can be (set theoretically) mapped into the coset space labeling the coherent states.* The peaking properties of coherent states will then give the peaking properties of the semi-classical states. Incidently, if in addition, the Hamiltonian of the system is the constraint itself, as is the case for canonical gravity in the cosmological context, then preservation of the peaking properties under time evolution is automatic.

It could be that for a particular choice of G, one may not get the desired correspondence. As long as there exist a choice of G commuting with the constraint and a choice of representation selected by the constraint which is labeled by the reduced phase space, we do have a class of semi-classical states.

Note that understanding the full $\mathcal{H}_{phy}$ and all the physical observables is also *not* essential for the identification of semi-classical states.

(2) Potential problems with group averaging when the constraint group is non-compact, can be bypassed as long as any regularization procedure adopted preserves the representation selection property (eqn. 4). As such one could translate the implications of group averaging as a condition on the rigging map and on the choice of Ω.

(3) Observe that the coherent states which give the simpler result depend on the group G which depends on the constraint. One does *not* start with a fixed set of coherent states in $\mathcal{H}_{kin}$ and define the physical ones via an explicit group averaging. In the examples discussed in the next section, the contrast will become apparent.

III. EXAMPLES

In this section we consider some simple toy models to illustrate the schematics discussed above. The chosen constraints are quadratic in phase space coordinates and momenta. Each of the case has some distinct feature. We will identify the group (of canonical transformations) commuting with the constraint, the reduced phase space and show the correspondence between the reduced phase space and the coherent space labels. In all the cases considered, the classical phase space is $\Gamma = R^4$ and we *choose* $\mathcal{H}_{kin}$ to be the usual Hilbert space of square integrable functions on R^2. For a more general and detailed analysis of the first two examples, please see [1].

A. The two dimensional harmonic oscillator constraint

The constraint can be written down as[1]

$$\phi = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2 - R^2) \tag{13}$$

where R^2 is positive. On using

$$q_i = \sqrt{\frac{\hbar}{2}}(a_i + a_i^\dagger), \qquad p_i = -\, i\sqrt{\frac{\hbar}{2}}(a_i - a_i^\dagger) \tag{14}$$

the constraint can be rewritten as

$$\phi = \hbar\,\left(a_1^\dagger\, a_1 + a_2^\dagger\, a_2 + 1 - \frac{R^2}{2\hbar}\right). \tag{15}$$

Among the physical observables permitted by this constraint are:

$$J_x = \frac{1}{2}\,(q_1\, q_2 + p_1\, p_2) = \frac{\hbar}{2}\,(a_1^\dagger\, a_2 + a_2^\dagger\, a_1), \tag{16}$$

$$J_y = \frac{1}{2}\,(q_1\, p_2 - q_2\, p_1) = \frac{i\,\hbar}{2}\,(a_2^\dagger\, a_1 - a_1^\dagger\, a_2), \tag{17}$$

$$J_z = \frac{1}{4}\,\left(q_1^2 + p_1^2 - q_2^2 - p_2^2\right) = \frac{\hbar}{2}\,(a_1^\dagger\, a_1 - a_2^\dagger\, a_2) \tag{18}$$

[1] In the following the index on coordinates is subscripted only for notational convenience.

which as can be easily seen to form the SU(2) Lie algebra,

$$[J_x, J_y] = i\hbar\, J_z, \quad [J_y, J_z] = i\hbar\, J_x, \quad [J_z, J_x] = i\hbar\, J_y. \tag{19}$$

The Casimir invariant (C_2) for the SU(2) group is $J^2 = J_x^2 + J_y^2 + J_z^2$ with eigenvalues $\hbar^2 j(j+1)$, which in our case becomes,

$$C_2 = \frac{\hbar^2}{4}\left[\frac{1}{\hbar^2}\left(\phi + \frac{R^2}{2}\right)^2 - 1\right] = \hbar^2 j(j+1). \tag{20}$$

When the constraint is imposed, it becomes,

$$C_2 = \frac{1}{4}\left[\frac{R^4}{4\hbar^2} - 1\right] = j(j+1). \tag{21}$$

The solution of j for the above equation is the representation of the physical coherent state which is selected by the group averaging procedure described in previous section.

Our next step is to identify the correspondence between the points on the reduced phase space and the points which serve as labels for the SU(2) coherent states. For that we rewrite the constraint as

$$\frac{q_1^2 + p_1^2}{R^2} + \frac{q_2^2 + p_2^2}{R^2} = 1 \tag{22}$$

which suggests a convenient parameterization,

$$q_1 = R\,\cos\theta\,\cos\varphi_1, \qquad p_1 = R\,\cos\theta\,\sin\varphi_1 \tag{23}$$

$$q_2 = R\,\sin\theta\,\cos\varphi_2, \qquad p_2 = R\,\sin\theta\,\sin\varphi_2. \tag{24}$$

The SU(2) group elements are parameterized as:

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \;\; \alpha \equiv \cos\frac{\mu}{2}\, e^{-i\,\nu_1},\; \beta \equiv -\,\sin\frac{\mu}{2}\, e^{-i\,\nu_2} \tag{25}$$

where $0 \le \mu \le 2\pi$ and $0 \le \nu_1, \nu_2 \le 2\,\pi$. Hence, the mapping between the constrained surface and the group manifold is given by $\mu/2 \longrightarrow \theta$, $-\nu_1 \longrightarrow \varphi_1$, $-\nu_2 \longrightarrow \varphi_2$.

The physical observables are related to the parameters on the constrained surface as

$$\frac{J_y}{J_x} = \tan(\varphi_2 - \varphi_1), \qquad J_z = \frac{R^2}{4}(2\,\cos^2\theta - 1) \tag{26}$$

Introducing $\varphi_\pm \equiv (\varphi_1 \pm \varphi_2)/2$, one sees that trajectories of the constraint just change φ_+. The reduced phase space is thus parameterized by θ, φ_-. Putting $\varphi_+ = 0$, we obtain the mapping between the reduced phase space and the coset space as $\mu/2 \longrightarrow \theta, -\nu \longrightarrow \varphi$, where we have redefined $\nu := \nu_2$ and $\varphi := \varphi_-$.

The SU(2) coherent states are labeled by ζ given in terms of the coset space labels as

$$\zeta = -\,\tan\frac{\mu}{2}\,e^{-i\nu} \tag{27}$$

and thus we get a mapping between the classical reduced phase space and the coherent states labels.

The SU(2) coherent states are the minimum uncertainty states of any pair of J_x, J_y and J_z,

$$|\zeta, j\rangle = \sum_{m=-j}^{j} \left[\frac{(2\,j)!}{(j+m)!\,(j-m)!}\right]^{1/2} (1+|\zeta|^2)^{-j}\,\zeta^{j+m}\,|j,m\rangle \tag{28}$$

with the resolution of identity,

$$\int d\mu(\zeta,j)|\zeta,j\rangle\,\langle\zeta,j| = 1, \qquad d\mu(\zeta,j) = \frac{2j+1}{\pi\,(1+|\zeta|^2)^2}\,d^2\zeta. \tag{29}$$

Since all the requirements of our general scheme are satisfied group averaging picks out the representation corresponding to $j = (-1 + R^2/2\hbar)/2$. The states of this representations are the ones which give the peaking with respect to the physical observables.

Remarks:

(1) The reducible representation of SU(2) in $\mathcal{H}_{kin}$ contains all allowed values of j. This can be inferred from the known spectrum of the two dimensional harmonic oscillator Hamiltonian. The fact that the j can only take discrete values, implies that in this case the group averaging produces physical states in $\mathcal{H}_{kin}$ itself. It also implies that if R does not have a value so as to select an allowed j, then there are *no* physical states either in $\mathcal{H}_{kin}$ or in a Ω^*. Thus R itself must take only a set of allowed values. For an analysis of this example from a different view point see also [8].

(2) In this case one could have used the standard coherent states of the oscillator and constructed physical states corresponding to these by direct explicit group averaging. The physical state comes out to be

$$|z_1, z_2\rangle = \int e^{-i\lambda\,(R^2/2\hbar - 1)}\,|z_1\,e^{i\lambda}, z_2\,e^{i\lambda}\rangle\,d\lambda \tag{30}$$

where

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle \tag{31}$$

and z is the eigenvalue of the annihilation operator $(a = (q + i\,p)/\sqrt{2\hbar})$.

In this case it turns out that $\langle J_i\rangle_{phy}$ are equal to their respective classical expressions, which is not surprising because the constraint under consideration is the harmonic oscillator Hamiltonian operator.

(3) The mapping between the points on the reduced phase space and the points which label the coherent states is important in our analysis. That establishes the correspondence between classical and quantum regimes. We will see that this is possible in all the cases considered.

B. The out of phase harmonic oscillator constraint

The constraint is of the form

$$
\begin{aligned}
\phi &= \frac{1}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2 - R^2) \\
&= \hbar \left(a_1^\dagger\, a_1 - a_2^\dagger\, a_2 - \frac{R^2}{2\hbar} \right) .
\end{aligned} \tag{32}
$$

The physical observables forming a Lie algebra in this case are

$$K_x = \frac{1}{2}\,(q_1\, p_2 + q_2\, p_1) = \frac{i\,\hbar}{2}\,(a_1^\dagger\, a_2^\dagger - a_1\, a_2), \tag{33}$$

$$K_y = \frac{1}{2}\,(q_1\, q_2 - p_1\, p_2) = \frac{\hbar}{2}\,(a_1^\dagger\, a_2^\dagger + a_1\, a_2), \tag{34}$$

$$K_z = \frac{1}{4}\,\left(q_1^2 + p_1^2 + q_2^2 + p_2^2\right) = \frac{\hbar}{2}\,(a_1^\dagger\, a_1 + a_2^\dagger\, a_2 + 1) \tag{35}$$

which form the SU(1,1) Lie algebra,

$$[K_x, K_y] = -i\,\hbar\, K_z, \quad [K_y, K_z] = i\,\hbar\, K_x, \quad [K_z, K_x] = i\,\hbar\, K_y. \tag{36}$$

The Casimir invariant for SU(1,1) group is:

$$C_2 = K_z^2 - K_x^2 - K_y^2. \tag{37}$$

Its eigenvalues are $\hbar^2 k(k-1)$, for discrete series and $\hbar^2(-\lambda^2 - \frac{1}{4})$, for continuous series. For the discrete series $k > 0$. For the continuous series to have a representation such that coherent states are labeled by a coset space one must have, $\lambda > 0$ for the principal continuous series and $-i\sigma/2 < \lambda < i\sigma/2, \lambda \neq 0$ for the supplementary continuous series. The Casimir invariant for this constraint turns out to be

$$C_2 = \frac{\hbar^2}{4}\left[(N_1 - N_2)^2 - 1\right] = \frac{\hbar^2}{4}\left[\frac{1}{\hbar^2}\left(\phi + \frac{R^2}{2}\right)^2 - 1\right] \tag{38}$$

where $N_i = a_i^\dagger\, a_i$. Then, it is easy to see that the only allowed series in this case is the discrete series and all members of this series occur in the reducible representation of SU(1,1) in $\mathcal{H}_{kin}$. Upon imposition of the constraint one gets,

$$C_2 = \frac{1}{4}\left[\frac{R^4}{4\hbar^2} - 1\right] = k(k-1) \quad . \tag{39}$$

The parameters on the constraint surface can be identified by rewriting the constraint as,

$$\frac{q_1^2+p_1^2}{R^2} - \frac{q_2^2+p_2^2}{R^2} = 1 \qquad (40)$$

which suggests,

$$q_1 = R\,\cosh\xi\,\cos\varphi_1, \qquad p_1 = R\,\cosh\xi\,\sin\varphi_1 \qquad (41)$$

$$q_2 = R\,\sinh\xi\,\cos\varphi_2, \qquad p_2 = R\,\sinh\xi\,\sin\varphi_2. \qquad (42)$$

Further, the SU(1,1) group elements are parameterized as,

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1, \quad \alpha \equiv \cosh\frac{\tau}{2}\,e^{-i\,\nu_1},\ \beta \equiv \sinh\frac{\tau}{2}\,e^{-i\,\nu_2}. \qquad (43)$$

where $\tau > 0$ and $0 \leq \nu_1, \nu_2 \leq 2\,\pi$. Hence, the mapping between the parameters on the constraint surface and those on group manifold is $\tau/2 \longrightarrow \xi,\ -\nu_1 \longrightarrow \varphi_1,\ -\nu_2 \longrightarrow \varphi_2$.

The physical observables are related to the parameters of the constraint surface as,

$$K_z = \frac{R^2}{4}\,(2\,\cosh^2\xi - 1), \qquad \frac{K_x}{K_y} = \tan(\varphi_1 + \varphi_2). \qquad (44)$$

Introducing $\varphi_\pm \equiv (\varphi_1 \pm \varphi_2)/2$, one sees that the trajectories of the constraint involve only changes in φ_-. The reduced phase space is thus parameterized by ξ, φ_+. Putting $\varphi_- = 0$, one obtains a mapping between the reduced phase space and the coset space (in this case it is the Lobachevsky plane) labeling the coherent states as, $\tau/2 \longrightarrow \xi, -\nu \longrightarrow \varphi$, where we have redefined $\nu := \nu_2$.

The SU(1,1) coherent space are labeled by points on the coset space as

$$\zeta = \tanh\frac{\tau}{2}\,e^{-i\,\nu} \quad , \quad |\zeta| < 1. \qquad (45)$$

With the above identifications we obtain the correspondence between the classical reduced phase space and the coherent states.

The SU(1,1) coherent states are also the minimum uncertainty states for the physical observables K_x and K_y, for the discrete series they are

$$|\zeta, k\rangle = (1 - |\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{\Gamma(n+1)\,\Gamma(2k)} \right]^{1/2} \zeta^n \, |k, n\rangle \tag{46}$$

with the following resolution of identity,

$$\int d\mu(\zeta, k) \, |\zeta, k\rangle \, \langle \zeta, k| = 1, \qquad d\mu(\zeta, k) = \frac{2k-1}{\pi \, (1 - |\zeta|^2)^2} \, d^2\zeta. \tag{47}$$

Group averaging picks out the representation[2] with $k = (1 + R^2/2\hbar)/2$.

Remarks:

(1) Since in this case coherent states correspond to a discrete series, the comments made in the context of SU(2) apply here also. Thus quantization is possible only if R is discrete.

(2) One could have used the standard coherent states in this case too. The physical state comes out to be

$$|z_1, z_2\rangle = \int e^{-i\lambda \, R^2/2\hbar} \, |z_1 \, e^{i\lambda}, z_2 \, e^{-i\lambda}\rangle \, d\lambda. \tag{48}$$

However, in this case one recovers the classical values for only K_x and K_y.

(3) In the case of SU(1,1) coherent states the resolution of identity is not defined for $k < 1/2$, however, this problem can be averted by resorting to the weak resolution of identity [9].

C. The two dimensional inverted oscillator

The constraint in this case is

---

[2]Another representation is picked out if $R^2 < 2\,\hbar$.

$$
\begin{aligned}
\phi &= \frac{1}{2}(-\,q_1^2 + p_1^2 - q_2^2 + p_2^2 - R^2) \\
&= -\frac{\hbar}{2}\left(a_1^{\dagger 2} + a_2^{\dagger 2} + a_1^2 + a_2^2 + \frac{R^2}{\hbar}\right).
\end{aligned} \tag{49}
$$

The physical observables corresponding to the above constraint are

$$
K_x = \frac{1}{4}\,\left(q_1^2 - p_1^2 - q_2^2 + p_2^2\right) = \frac{\hbar}{4}\,(a_1^{\dagger 2} - a_2^{\dagger 2} + a_1^2 - a_2^2), \tag{50}
$$

$$
K_y = \frac{1}{2}\,(q_1\,q_2 - p_1\,p_2) = \frac{\hbar}{2}\,(a_1^\dagger\,a_2^\dagger + a_1\,a_2) \tag{51}
$$

$$
K_z = \frac{1}{2}\,(q_1\,p_2 - q_2\,p_1) = \frac{i\,\hbar}{2}\,(a_2^\dagger\,a_1 - a_1^\dagger\,a_2) \tag{52}
$$

which also form an SU(1,1) algebra. The Casimir invariant for this case is

$$
C_2 = -\frac{\hbar^2}{4}\,\left[\frac{1}{\hbar^2}\left(\phi + \frac{R^2}{2}\right)^2 + 1\right]. \tag{53}
$$

The representations allowed in this case are corresponding to the principal continuous series. Upon imposition of the constraint, the Casimir becomes

$$
C_2 = -\frac{1}{4}\,\left[\frac{R^4}{4\hbar^2} + 1\right] = -(\lambda^2 + \frac{1}{4}). \tag{54}
$$

To get the identification with SU(1,1) group manifold, we rewrite the constraint as,

$$
\frac{p_1^2 + p_2^2}{R^2} - \frac{q_1^2 + q_2^2}{R^2} = 1 \tag{55}
$$

which suggests a parameterization,

$$
p_1 = R\,\cosh\mu\,\cos\gamma_1, \qquad p_2 = R\,\cosh\mu\,\sin\gamma_1 \tag{56}
$$

$$
q_1 = R\,\sinh\mu\,\sin\gamma_2, \qquad q_2 = R\,\sinh\mu\,\cos\gamma_2. \tag{57}
$$

Hence, the identification between the group manifold and coset labels for SU(1,1) coherent states is $\tau/2 \longrightarrow \mu$ and $-\nu \longrightarrow (\gamma_1 + \gamma_2)$.

Note that on the constraint surface $(-\,q_1^2 + p_1^2)$ and $(q_2^2 - p_2^2)$ are constant. Furthermore, we can rewrite K_x and $K_z^2 - K_y^2$ as sum and product of these,

$$K_x = -\frac{1}{4}\left[(-\,q_1^2 + p_1^2) + (q_2^2 - p_2^2)\right], \qquad K_z^2 - K_y^2 = \frac{1}{4}\left[(-q_1^2 + p_1^2)\,(q_2^2 - p_2^2)\right]. \quad (58)$$

In order to get the mapping between the reduced phase space and the coset space we write the constraint in the form which will give natural parameters for the constraint surface,

$$\frac{(-\,q_1^2 + p_1^2)}{R^2} - \frac{(q_2^2 - p_2^2)}{R^2} = 1 \qquad (59)$$

which leads to three possibilities:

(i) $(-\,q_1^2 + p_1^2) > 0$ and $(q_2^2 - p_2^2) > 0$,

(ii) $(-\,q_1^2 + p_1^2) < 0$ and $(q_2^2 - p_2^2) < 0$,

(iii) $(-\,q_1^2 + p_1^2) > 0$ but $(q_2^2 - p_2^2) < 0$.

The fourth possibility of $(-\,q_1^2 + p_1^2) < 0$ and $(q_2^2 - p_2^2) > 0$ is ruled out because R^2 is always positive.

Remark:

The three cases are mutually exclusive. Each of these constitutes a connected component of the reduced phase space. One can restrict one self to any one of these. In the following, all three are considered but it is to be remembered that only one is relevant at a time.

For each of these cases there is a natural parameterization in terms of (ξ, η_1, η_2),

case (i):

$$q_1 = R\,\cosh\xi\,\sinh\eta_1, \qquad p_1 = R\,\cosh\xi\,\cosh\eta_1 \qquad (60)$$

$$q_2 = R\,\sinh\xi\,\cosh\eta_2, \qquad p_2 = R\,\sinh\xi\,\sinh\eta_2, \qquad (61)$$

case (ii):

$$q_1 = R\,\sinh\xi\,\cosh\eta_1, \qquad p_1 = R\,\sinh\xi\,\sinh\eta_1 \tag{62}$$

$$q_2 = R\,\cosh\xi\,\sinh\eta_2, \qquad p_2 = R\,\cosh\xi\,\cosh\eta_2, \tag{63}$$

and case (iii):

$$q_1 = R\,\mathrm{sech}\,\xi\,\sinh\eta_1, \qquad p_1 = R\,\mathrm{sech}\,\xi\,\cosh\eta_1 \tag{64}$$

$$q_2 = R\,\tanh\xi\,\sinh\eta_2, \qquad p_2 = R\,\tanh\xi\,\cosh\eta_2. \tag{65}$$

In case (iii), note that the choice of the use of $\sin\theta$ and $\cos\theta$ instead of $\mathrm{sech}\,\xi$ and $\tanh\xi$ is not suitable because then we would have to omit points $\theta = (0,\pi)$ and/or $(\pi/2, 3\pi/2)$. The reduced phase space is hence made up of disconnected parts.

The physical observables can be expressed in terms of (ξ,η_1,η_2) and one finds out that the parameters on the reduced phase space are ξ and $(\eta_1-\eta_2)$. For example, in the case (i) we get,

$$K_x = \frac{R^2}{4}\,(1 - 2\,\cosh^2\xi), \qquad \frac{K_y}{K_z} = \tanh(\eta_2-\eta_1). \tag{66}$$

With the parameterization of the group manifold, coset space and reduced phase space given, it can be shown that there is a 1-1 and onto mapping between the coset space and the reduced phase space. We will show this for case (i), other cases follow similarly.

We first define $\eta_\pm = (\eta_1 \pm \eta_2)/2$. Further, note that $\dot\eta_+ = 1$ and $\dot\eta_- = 0$. Without any loss of generality we can choose $\eta_+ = 0$ to get to the reduced phase space. With these substitutions in eqs(60,61) and then equating the resultant set with eqs.(56,57), we get

$$q_1 = R\,\cosh\xi\,\sinh\eta = R\,\sinh\mu\,\sin\gamma_2 \tag{67}$$

$$p_1 = R\,\cosh\xi\,\cosh\eta = R\,\cosh\mu\,\cos\gamma_1 \tag{68}$$

$$q_2 = R\,\sinh\eta\,\cosh\eta = R\,\sinh\mu\,\cos\gamma_2 \tag{69}$$

$$p_2 = -R\,\sinh\xi\,\sinh\eta = R\,\cosh\mu\,\sin\gamma_1 \tag{70}$$

where $\eta = \eta_-$. The above set of equations leads to,

$$\sinh 2\mu \, \sin(\gamma_1 + \gamma_2) = \sinh 2\eta, \;\; \sinh 2\mu \, \cos(\gamma_1 + \gamma_2) = \sinh 2\xi \, \cosh 2\eta. \tag{71}$$

With our new definitions (ξ, η) parameterize the reduced phase space and as noted earlier μ and $(\gamma_1 + \gamma_2)$ are related to the coset space labels for the SU(1,1) coherent states ($\tau/2 \longrightarrow \mu$ and $-\nu \longrightarrow (\gamma_1 + \gamma_2)$). Hence, the above equation gives us a mapping between the reduced phase space and the coset space. That the mapping is 1-1 and onto is easy to check, since given either of $(\mu, (\gamma_1 + \gamma_2))$ or (ξ, η), one can determine other uniquely. This can be shown similarly for other two cases.

Hence, reduced phase space is made up of the three copies of the same space which is equivalent to the coset space. Which copy is to be picked out is to be *predetermined* classically as remarked above.

Since, in this case coherent states corresponding to principal continuous series are allowed, they are given by expansions in orthonormal basis with expansion functions as the eigenfunctions of Laplace-Beltrami operator for the Lobachevsky plane. We refer the reader to Perelomov's monograph [7] for more details. The representation which is picked out by group averaging is $\lambda = R^2/4\hbar$.

Remark:

As in the previous example, we have the same non-compact group here, but now the representations in $\mathcal{H}_{kin}$ belong to the principal continuous series. The use of the standard coherent states leads to messy algebra in obtaining the group averaged states, obscuring any correspondence between quantum and classical states. However, the natural choice of coherent states of the invariance group simplifies the computations considerably.

IV. DISCUSSION

We have looked at the simplest (smallest dimensional) non-trivial possibilities for a constrained system. Our examples involve both compact and non-compact semi-simple groups. In all cases, the symmetry group was the group of linear canonical transformations leaving the constrained surface invariant. The constrained surface itself was a group manifold. Dimensions of the reduced phase spaces and the coset spaces labeling the coherent states also coincided. This need not always be the case. For instance if one considers the first example generalized to N dimensions, one encounters SU(N) as the symmetry group. The constrained surface is S^{2N-1} which is not a group manifold for $N > 2$. If one uses the SU(N) coherent states given in [10], then these are labeled by (2N -1) dimensional coset space ($\frac{SU(N)}{SU(N-1)} \sim S^{2N-1}$) while the reduced phase space is of dimension (2N -2). Semi-classical states then form a *proper subset* of the coherent states. The schematics of section II of course applies to the coherent states. This is a case in which the reduced phase space is mapped *in to* the coset space labeling the coherent states.

Another example which can be commented upon is the case with $\Gamma = R^{2N}$ and constraint being $q_N = 0$ (say). The choice of G in this case would be the Heisenberg-Weyl group in the $(2N-2)$ phase space variables. The reduced phase space is trivially R^{2N-2} and the standard coherent states will suffice. The physical inner product among these is automatically restricted to the $(N-1)$ dimensional Lebesgue measure.

While there are very many interesting examples that can be analyzed, eg. the case of a free relativistic particle constraint, $P_0^2 - \vec{P}\cdot\vec{P} = m^2$ and its non-relativistic cousins, it is not our aim to do so here. We have been primarily motivated by the quantum geometry context wherein the issue of semi-classical states is being addressed. One issue in this regards is the need to identify semi-classical states in $\mathcal{H}_{phy}$ on the one hand and having good control only over $\mathcal{H}_{kin}$ on the other hand. Our schematic derivation addresses this issue though with further assumptions made.

The toy models considered in this work bring out various points one should keep in mind while obtaining physical semi-classical states through refined algebraic quantization.

(1) The use of coherent states for the invariance group allows us to establish the result that the inner product, expectation values of and quantum fluctuations in physical observables are same for the kinematical and physical states, if kinematical states are chosen as the group coherent states with a specific representation. We have explicitly demonstrated this in the three non trivial examples and it is apparent that this can be done whenever we have an invariance group and suitable coherent states available.

(2) The mapping between the reduced phase space and the coset space labels which define the coherent states is generally non trivial. This has been seen in the case of an inverted oscillator. This is a point to be kept in mind because apart from the simple problems in most cases one is not likely to know the reduced phase space sufficiently explicitly.

(3) While the availability of group averaging as a method of choosing η mapping is tempting to suggest that one could start with a notion of semi-classical states in $\mathcal{H}_{kin}$ and group average these to *define* semi-classical states in $\mathcal{H}_{phy}$, it can be extremely cumbersome. In our examples this corresponds to choosing the standard coherent states as starting states and implementing group averaging. We have seen how clumsy this is. Therefore an important lesson is that one should not over-emphasize the use of standard coherent states (or coherent states natural to the structure of $\mathcal{H}_{kin}$) in all situations. While the standard coherent states are a natural choice for harmonic oscillator con-

straint they prove quite unsuitable for the inverted oscillator constraint.

We should emphasize that we have *not* taken the examples as illustration of the algebraic quantization scheme. We are also *not* using coherent states to quantize a constrained theory i.e. construct the full physical state space. We are using coherent states with a limited purpose as a means to identify candidate *semi-classical* states. Towards this end, particular choice of $\mathcal{H}_{kin}$ and precise details of group averaging are not too critical.

Our scheme also has some limitations. It relies on the availability of a suitable Lie group (as opposed to only a Lie algebra) commuting with the constraint. Even if a group is available, it relies on availability of suitable coherent states. Lastly it relies on finding or demonstrating existence of a suitable map between the label space for the coherent states and the physical reduced phase space. Typically coherent states are labeled by a (connected) coset space i.e. a connected homogeneous space whereas a reduced phase space need not be so. It is however a conceivable possibility that reduced phase space may map in to only a subset of coherent states. Each of these aspects needs to be explored and generalized further to get to interesting realistic systems.

However, if these limitations can be circumvented then one has a possibility of getting the semi-classical states at least without having to know the full physical space.

Acknowledgements: PS is supported by CSIR grant number: 2 - 34/98(ii)E.U-II. He thanks the Institute of Mathematical Sciences for warm hospitality and a pleasant stay during which this work was completed.

REFERENCES

[1] A. Ashtekar, R. S. Tate, J. Math. Phy. **35** (1994) 6434, gr-qc/9405073.

[2] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, J. Math. Phy. **36** (1995) 6456, gr-qc/9504018.

[3] D. Marolf in *Symplectic Singularities and Geometry of Gauge Fields*, Banach Center publications, Polish academy of Science, Institute of Mathematics, Warsaw, 1997, gr-qc/9508015.

[4] D. Marolf, *Proceedings of the 9th Marcel Grossmann Conference*, Rome, 2000, gr-qc/0011112.

[5] D. Giulini, D. Marolf, Class. Quant. Grav. **16** (1999) 2479, gr-qc/9812024; Class. Quant. Grav. **16** (1999) 2489, gr-qc/9902045.

[6] D. Giulini, Nucl. Phys. Proc. Suppl. **88** (2000) 385, gr-qc/0003040.

[7] A. Perelomov, *Generalized Coherent States and Their Applications,* Springer-Verlag, Berlin (1986).

[8] R. Vathsan, Jour. Math. Phys. **37** (1996) 1713; Erratum -ibid **37** (1996) 6590, hep-th/9507066.

[9] C. Brif, A. Vourdas, A. Mann, J. Phys. **A 29** (1996) 5873, quant-ph/9607022.

[10] K. Nemoto, J. Phys. **A 33** (2000) 3493. , quant-ph/0004087.